Numerical Integration

Ching-Han Chen
I-Shou University
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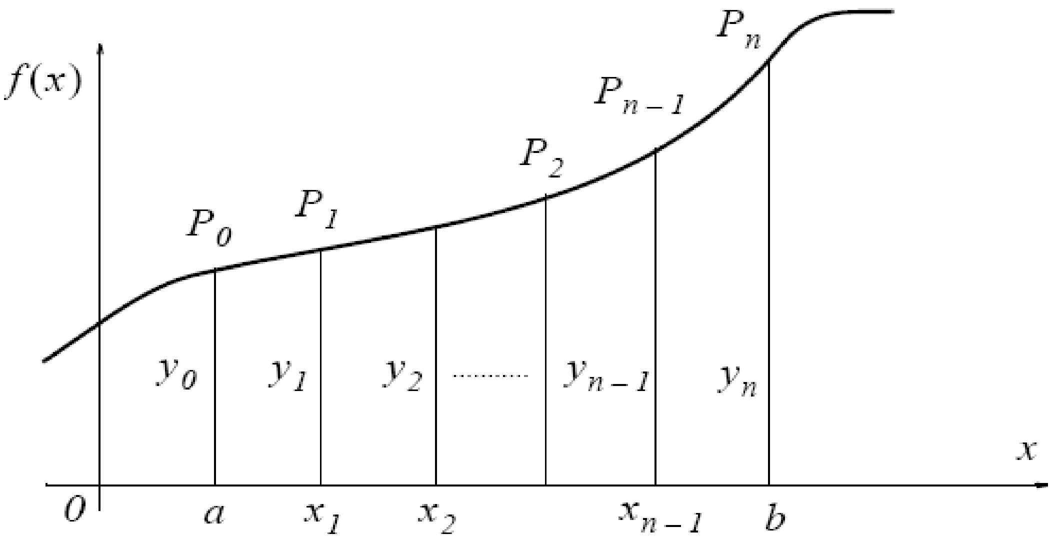
Integration

For *a linear function y* = *f* (*x*), we divide the the interval *a*≤*x*≤*b* into n subintervals, each of length $\Delta x = \frac{b-a}{n}$



x1 = *a* + Δ*x*,

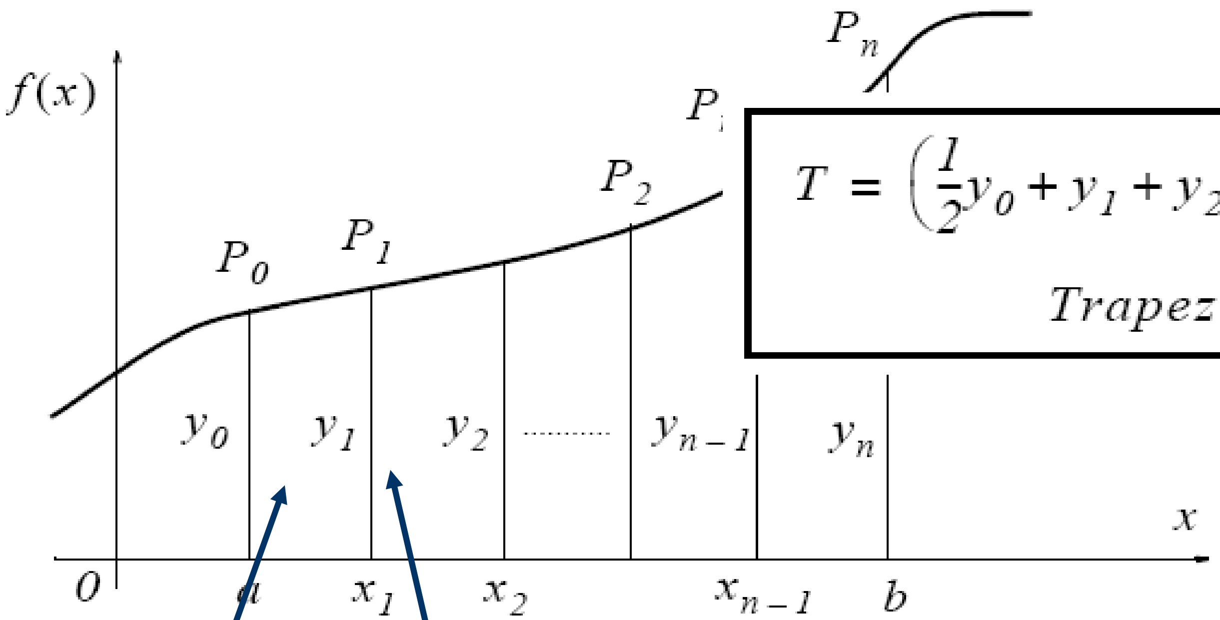
x2 =*a*+2Δ*x*,

…,

xn–*1* =*a*+(*n*–*1*)Δ*x*

$$\int_a^b f(x)dx = \int_a^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \ldots + \int_{x_{n-1}}^{b} f(x)dx = \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} f(x)dx$$

Trapezoid Approximation



$$T = \left(\frac{1}{2}y_0 + y_1 + y_2 + \ldots + y_{n-1} + \frac{1}{2}y_n\right)\Delta x$$

Trapezoidal Rule

$\frac{1}{2}(y_0 + y_1)\Delta x$ $\frac{1}{2}(y_1 + y_2)\Delta x$

Trapezoid Approximation

$$T = \left(\frac{1}{2}y_0 + y_1 + y_2 + \ldots + y_{n-1} + \frac{1}{2}y_n\right)\Delta x$$

Trapezoidal Rule

with y = f (x)

```
interval = (max-min) / n;
sum=0;
for (i=1; i<n; i++)                          // sum the midpoints
{
   x    = min + interval * i;
   sum = sum + f(x)*interval;
}
sum += 0.5 *(f(min) + f(max)) * interval;  // add the endpoints
```

Ex1.

Using the trapezoid rule with *n=4,* , estimate the value of the integral

$$\int_1^2 x^2\,dx$$

The exact value of this integral is $\int_1^2 x^2\,dx = \left.\frac{x^3}{3}\right|_1^2 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3} = 2.33333$

Using the trapezoid rule approximation to compute the integral

$$x_0 = a = 1$$

$$x_n = b = 2$$

$$n = 4$$

$$\Delta x = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4}$$

Ans. 2.34375

Ex2.

Using the trapezoidal rule with *n*=10, , estimate the value of the integral

$$\int_1^2 x^2\,dx$$

Ex3. Compute the energy dissipated

The *i-v* relation of a non-linear electrical device is given by

$$i(t) = 0.1(e^{0.2v(t)} - 1)$$ *with v*(*t*) = *sin3t*

The instantaneous power *p*(*t*) will be

$$p(t) = v(t)i(t) = 0.1 \sin 3t(e^{0.2 \sin 3t} - 1)$$

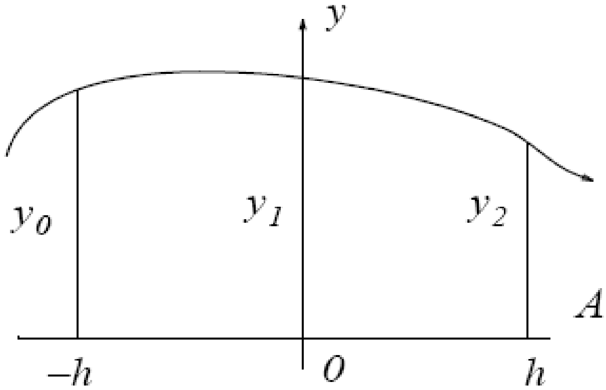
The energy *W*(*t0*, *t1*) dissipated in this device from *t0 = 0 to t1 = 10*

$$W(t_0, t_1) = \int_{t_0}^{t_1} p(t)dt = 0.1\int_0^{10\text{ s}} \sin 3t(e^{0.2 \sin 3t} - 1)dt$$

Ans: 0.1013

Simpson's Rule

For a parabola curve $y = \alpha x^2 + \beta x + \gamma$



The area under this curve for the interval $-h \le x \le h$ is

$$Area\Big|_{-h}^{h} = \int_{-h}^{h} (\alpha x^2 + \beta x + \gamma)dx = \frac{\alpha x^3}{3} + \frac{\beta x^2}{2} + \gamma x \Big|_{-h}^{h}$$

$$= \frac{\alpha h^3}{3} + \frac{\beta h^2}{2} + \gamma h - \left(-\frac{\alpha h^3}{3} + \frac{\beta h^2}{2} - \gamma h\right) = \frac{2\alpha h^3}{3} + 2\gamma h$$

$$= \frac{1}{3}h(2\alpha h^3 + 6\gamma)$$

Simpson's rule:

$$Area\Big|_{-h}^{h} = \frac{1}{3}h(y_0 + 4y_1 + y_2)$$

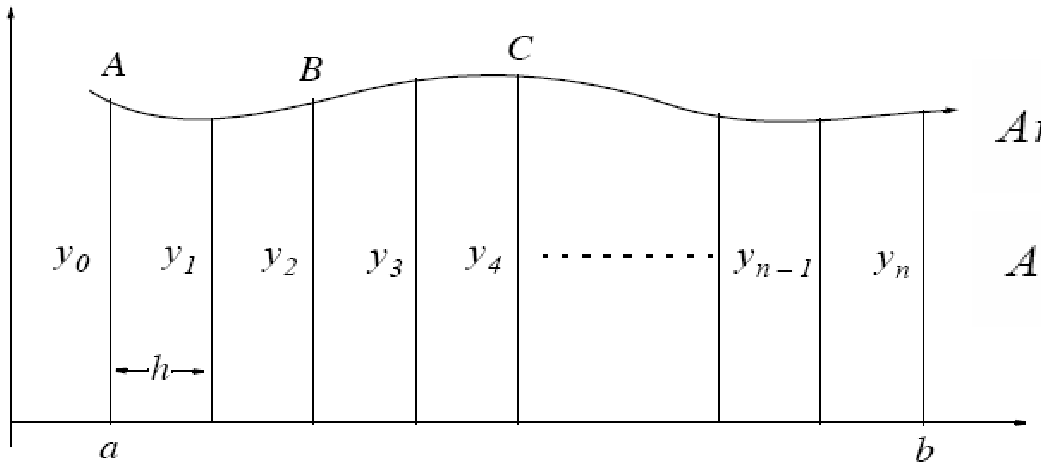
and

$$y_0 = \alpha h^2 - \beta h + \gamma \qquad (a)$$

$$y_1 = \gamma \qquad (b)$$

$$y_2 = \alpha h^2 + \beta h + \gamma \qquad (c)$$

Simpson’s rule of integration



$$Area\big|_{AB} = \frac{1}{3}h(y_0 + 4y_1 + y_2)$$

$$Area\big|_{BC} = \frac{1}{3}h(y_2 + 4y_3 + y_4)$$

$$h = \frac{b-a}{n} \qquad n = even$$

$$Area = \frac{1}{3}h(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \ldots + 2y_{n-2} + 4y_{n-1} + y_n)$$

Simpson’s Rule of Numerical Integration

Simpson's rule of integration

```
float interval, sum, x;
interval = ((max -min) /n);
sum=0;
for (i=1; i<n; i=i+2)         //loop for odd points
{
    x = min + interval * i;
    sum += 4 * f(x);
}
for (i=2; i<n; i=i+2)          // loop for even points
{
   x = min+interval * i;
   sum += 2 * f(x);
}
sum +=  f(min) + f(max);   // add first and last value
sum *= interval/3.;        // then multilpy by interval
```

Ex4.

Using the Simpson's rule with n=10, estimate the value of the integral

$$y = f(x) = \int_0^2 e^{-x^2}\, dx$$

Ans. 0.8820

Ex5.

Using respectively the Trapezoid rule and Simpson's rule *to* estimate the value of the integral. Find resonable n for Trapezoid rule while Simpson's rule use n=8.

$$\int_{1}^{2} \frac{1}{x} dx$$

And plot the curve of approximated integral vale respect to different n, n=2,4,6,…, 100.

Hint : the analytical value of this definite integral is the ***natural log***
ln= 0.6931